Many pulses heteroclinic orbits with a Melnikov method and chaotic dynamics of a parametrically and externally excited thin plate

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Abstract The multi-pulse heteroclinic orbits with a Melnikov method and chaotic dynamics in a parametrically and externally excited thin plate are studied in this paper for the first time. The thin plate is subjected to transversal and in-plane excitations, simultaneously. The formulas of the thin plate are derived from the von Kármán equation and Galerkin’s method. The method of multiple scales is used to find the averaged equation. The theory of normal form, based on the averaged equation, is used to obtain the explicit expressions of normal form associated with a double zero and a pair of purely imaginary eigenvalues from the Maple program. Based on normal form obtained above, an extension of the Melnikov method is utilized to analyze the multi-pulse global bifurcations and chaotic dynamics in a parametrically and externally excited thin plate. The global dynamics analysis indicates that there exist the multi-pulse jumping orbits in the perturbed phase space of the averaged equations for a parametrically and externally excited thin plate. These results show that the chaotic motions of the multi-pulse Shilnikov type can occur in a parametrically and externally excited thin plate. Numerical simulations are given to verify the analytical predictions. It is also found from the results of numerical simulation that the multi-pulse Shilnikov type orbits exist in a parametrically and externally excited thin plate.

1 Introduction

Engineering researchers have paid much attention to studying the nonlinear oscillations of thin plates in the case of large deformation. With the use of thin plate in shuttles and large space stations, nonlinear dynamics, bifurcations, and the chaos of thin plates have become more and more important. In the past decade, researchers have made a number of studies into nonlinear oscillations, bifurcation and the chaos of thin plate. Yang and Sethna [1] used an averaging method to study the local and global bifurcations in parametrically excited, nearly square plates. The results obtained indicated that heteroclinic loops exist and Smale horse and chaotic motions can occur. Based on the studies in [1], Feng and Sethna [2] made use of the global perturbation method to study further the bifurcations and chaotic dynamics of a thin plate under parametric excitation, and obtained the conditions in which Shilnikov-type homoclinic orbits and chaos can occur. Nayfeh and Vakakis [3] used the method of multiple scales to study sub-harmonic traveling waves of thin, axi-symmetric, geometrically nonlinear circular plates and found the nonlinear interactions of pairs of modes with coincident linearized natural frequencies. Tian et al. [4,5] used the averaging method and Melnikov technique to study local, global bifurcations and chaos of a two-degree-of-freedom shallow arch subjected to simple harmonic excitation for case of 1:2 and 1:1 internal resonances. The global bifurcations and chaotic dynamics were investigated by Zhang et al. [6] and Zhang [7] for both parametrically-externally excited and parametrically excited simply supported rectangular thin plates.

The global bifurcations and chaotic dynamics of high-dimensional nonlinear systems have been at the forefront of nonlinear dynamics for the last two decades. Some new phenomena on the global bifurcations and chaotic dynamics are discovered in high-dimensional nonlinear systems, such as the multi-pulse Shilnikov orbits. However, due to lack of analytical tools to study global bifurcations and chaotic dynamics for high-dimensional nonlinear systems, it is extremely challenging to develop theories of global bifurcations and chaotic dynamics for high-dimensional nonlinear systems and to give systematic applications to engineering problems. Despite the challenge, certain progress has been achieved in this field in the past two decades.

Wiggins [8] divided four-dimensional perturbed Hamiltonian systems into three types and utilized the Melnikov method to investigate the global bifurcations and chaotic dynamics for these three basic systems. Based on the study given by Wiggins [8], Kovacic and Wiggins [9] developed a new global perturbation technique which may be used to detect the Shilnikov type single-pulse homoclinic and heteroclinic orbits in four-dimensional autonomous ordinary differential equations and gave an
application to the forced and damped sine-Gordon equation. Later on, Kaper and Kovacic [10] employed a modified Melnikov method to study the existence of several types of multi-bump homoclinic orbits to resonance bands for completely integral Hamiltonian systems subjected to small amplitude Hamiltonian and damped perturbations. Camassa et al. [11] extended the Melnikov method to investigate multi-pulse jumping of homoclinic and heteroclinic orbits in a class of perturbed Hamiltonian systems.

Besides the aforementioned researches on theories of global bifurcations and chaotic dynamics in high-dimensional nonlinear systems, it is worth mentioning researches on applying the developed theories to engineering applications. Malhotra et al. [12] used the energy-phase method to investigate multi-pulse homoclinic orbits and chaotic dynamics in the motion of flexible spinning discs. The extended subharmonic Melnikov method and the modified homoclinic Melnikov method were employed by Yagasaki [13] to analyze periodic orbits and homoclinic motions in periodically forced, weakly coupled oscillators with the perturbations. Furthermore, Zhang and Li [14] employed the global perturbation approach to investigate the global bifurcations and chaotic dynamics for a two-degree-of-freedom nonlinear vibration absorber. Recently, Zhang and Tang [15] studied the global bifurcations and chaotic dynamics of the suspended elastic cable to small tangential vibration of one support which causes simultaneously the parametric excitation of out-plane motion and the parametric and external excitations of in-plane motion.

The studies of this paper focus on the multi-pulse Shilnikov orbits and chaotic dynamics for a simply supported at the fore-edge, rectangular thin plate subjected to transversal and in-plane excitations simultaneously. We only study the case of 1:2 internal resonance and primary parametric resonance fundamental parametric resonance. Firstly, the governing equations of the rectangular thin plate are derived based on the von Karman equation and the equations of motion with two-degree-of-freedom under combined parametrical and external excitations can be obtained by using Galerkin’s method, respectively. Then the method of multiple scales is utilized to obtain the averaged equations from the original non-autonomous system. From the averaged equation, the theory of normal form is used to find the explicit formulas of the normal form. Finally, the extended Melnikov method presented by Camassa and Kovacic [11] is employed to analyze the multi-pulse heteroclinic orbits and chaotic dynamics for the simply supported rectangular thin plate. The analysis indicates that there exist the multi-pulse jumping orbits in the perturbed phase space for the averaged equations. The results of numerical simulation also show that the chaotic motion can occur in the nonlinear oscillations of the simply supported rectangular thin plate subjected to transversal and in-plane excitations, which verifies the analytical prediction. The multi-pulse orbits are discovered from the results of numerical simulation.

2. Equations of Motion and Perturbation Analysis

We consider a simply supported at the fore-edge, rectangular thin plate, where the edge lengths are $a$ and $b$ and thickness is $h$, respectively. The thin plate is subjected to transversal and in-plane excitation, simultaneously. We establish a Cartesian coordinate system, shown in Figure 1, such that the coordinate $Oxy$ is located in the middle surface of thin plate. It is assumed that $u, v$ and $w$ represent the displacements of a point in the middle plane of the thin plate in the $x, y$ and $z$ directions, respectively. The excitation in-plane of the thin plate may be expressed in the form $p = p_0 - p_1 \cos \Omega t$. From the von Karman-type equations for the thin plate [18], we obtain the equations of motion for the rectangular thin plate as follows:

$$
DV^4 w + \rho \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 \phi}{\partial x^2} + 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 \phi}{\partial x \partial y} + \mu \frac{\partial w}{\partial t} = F(x, y) \cos \Omega t, \quad (1)
$$

$$
\nabla^4 \phi = Eh \left[ \frac{\partial^2 w}{\partial \partial x} \right]^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}, \quad (2)
$$

where $\rho$ is the density of thin plate, $D = Eh^3/(12(1 - \nu^2))$ is the bending rigidity, $E$ is Young’s modulus, $\nu$ is Poisson’s ratio, $\phi$ is the stress function, and $\mu$ is the damping coefficient.

We mainly consider the nonlinear oscillations of a thin plate in the first two modes. By means of Galerkin’s method, we obtain the dimensionless equations of motion as follows:
\[ \ddot{x}_1 + \epsilon \mu \dot{x}_1 + (\omega_1^2 + 2\sigma f_1 \cos \Omega_2 t) x_1 + \epsilon(\alpha_1 x_1^3 + \alpha_2 x_1 x_2^2) = \epsilon F_1 \cos \Omega_2 t, \]
\[ \ddot{x}_2 + \epsilon \mu \dot{x}_2 + (\omega_2^2 + 2\sigma f_2 \cos \Omega_2 t) x_2 + \epsilon(\beta_1 x_2^3 + \beta_2 x_1 x_2^2) = \epsilon F_2 \cos \Omega_2 t, \] (3)

\[ F(x, y) \cos \Omega_2 t \]

\[ p = p_0 - p_1 \cos \Omega_2 t \]

Figure 1 The model of a rectangular thin plate and the coordinate system.

We use the method of multiple scales [19] to find the uniform solutions of equation (3) in the following form:

\[ x_n(t, \epsilon) = x_{n0}(T_0, T_1) + \epsilon x_{n1}(T_0, T_1) + \cdots, n = 1, 2, \] (4)

where \( T_0 = t, T_1 = \epsilon t \). Then, we have the differential operators

\[ \frac{d}{dt} = \frac{\partial}{\partial T_0} \frac{T_0}{\partial t} + \frac{\partial}{\partial T_1} \frac{T_1}{\partial t} + \cdots = D_0 + \epsilon D_1 + \cdots, \] (5)

\[ \frac{d^2}{dt^2} = (D_0 + \epsilon D_1 + \cdots)^2 = D_0^2 + 2\epsilon D_0 D_1 + \cdots, \] (6)

where \( D_k = \frac{\partial}{\partial T_k}, k = 0, 1, \).

We only study the case of 1:2 internal resonance and primary parametric resonance-fundamental parametric resonance. In this resonant case there are the following relations:

\[ \omega_1^2 = \frac{1}{4} \Omega_2^2 + \epsilon \sigma_1, \omega_2^2 = \Omega_2^2 + \epsilon \sigma_2, \Omega_1 = \Omega_2, \] (7)

where \( \sigma_1 \) and \( \sigma_2 \) are the two detuning parameters. For convenience of the study, we let \( \Omega_1 = \Omega_2 = 2 \).

Substituting equations (4-7) into equations (3) and equating the coefficients of like power of \( \epsilon \) in left side and right side of the equations, we obtain the differential equations. Separating the real and imaginary parts, the averaged equations in the Cartesian form are obtained as follows

\[ \frac{dx_1}{dT_1} = -\frac{1}{2} \mu x_1 - \frac{1}{2} (\sigma_1 - f_1) x_1 - \frac{3}{2} \alpha_1 x_2 (x_1^3 + x_2^2) - \alpha_2 x_2 (x_3^2 + x_4^2), \] (8a)

\[ \frac{dx_2}{dT_1} = \frac{1}{2} (\sigma_1 + f_1) x_1 - \frac{1}{2} \mu x_2 + \frac{3}{2} \alpha_1 x_1 (x_1^3 + x_2^2) + \alpha_2 x_1 (x_3^2 + x_4^2), \] (8b)

\[ \frac{dx_3}{dT_1} = -\frac{1}{2} \mu x_3 - \frac{1}{4} \sigma_2 x_4 - \frac{3}{4} \beta_1 x_2 (x_3^2 + x_2^2) - \frac{1}{2} \beta_2 x_4 (x_3^2 + x_4^2), \] (8c)

\[ \frac{dx_4}{dT_1} = -\frac{1}{8} F_2 + \frac{1}{4} \sigma_2 x_3 - \frac{1}{2} \mu x_4 + \frac{3}{4} \beta_1 x_3 (x_3^2 + x_4^2) + \frac{1}{2} \beta_2 x_3 (x_3^2 + x_4^2). \] (8d)

In the next section, we will give normal form of averaged equation (8) for the nonlinear oscillations of the simply supported rectangular thin plate under combined parametrical and external excitations.

3. Computation of Normal Form

In order to conveniently analyze the multi-pulse Shilnikov type orbits and chaotic dynamics for the nonlinear oscillations of the simply supported rectangular thin plate subjected to transversal and in-plane excitations, we need to reduce averaged equation (8) to a simpler normal form. It is seen that there are \( Z_2 \oplus Z_2 \) and \( D_4 \) symmetries in averaged equation (8) without the parameters. Therefore, these symmetries are also held in normal form.
Take into account the exciting amplitude $F_2$ as a perturbation parameter. Amplitude $F_2$ can be considered as an unfolding parameter when the multi-pulse Shilnikov type orbits are investigated. Obviously, when we do not consider the perturbation parameter, equation (8) becomes

$$
\frac{dx_1}{dT} = -\frac{1}{2} \mu_1 - \frac{1}{2} (\sigma_1 - f_1) x_2 - \frac{3}{2} \alpha_1 x_2 (x_1^2 + x_2^2) - \alpha_2 x_2 (x_3^2 + x_4^2), \quad (9a)
$$

$$
\frac{dx_2}{dT} = \frac{1}{2} (\sigma_1 + f_1) x_1 - \frac{1}{2} \mu_2 + \frac{3}{2} \alpha_1 x_1 (x_1^2 + x_2^2) + \alpha_2 x_1 (x_3^2 + x_4^2), \quad (9b)
$$

$$
\frac{dx_3}{dT} = -\frac{1}{2} \mu_3 - \frac{1}{4} \sigma_2 x_4 - \frac{3}{4} \beta_1 x_4 (x_3^2 + x_4^2) - \frac{1}{2} \beta_2 x_4 (x_1^2 + x_2^2), \quad (9c)
$$

$$
\frac{dx_4}{dT} = \frac{1}{4} \sigma_2 x_3 - \frac{1}{2} \mu_4 + \frac{3}{4} \beta_1 x_3 (x_3^2 + x_4^2) + \frac{1}{2} \beta_2 x_3 (x_1^2 + x_2^2). \quad (9d)
$$

Executing the Maple program given by Zhang et al. [20], the 3-order normal form of system (9) is obtained as

$$
\dot{y}_1 = y_2, \quad (10a)
$$

$$
\dot{y}_2 = \frac{3}{2} \alpha_1 y_1^3 + \alpha_2 y_1 y_2^2 + \alpha_2 y_1 y_4^2, \quad (10b)
$$

$$
\dot{y}_3 = -\frac{1}{4} \sigma_2 y_4 - \frac{3}{4} \beta_1 y_4 (x_3^2 + x_4^2) - \frac{1}{2} \beta_2 y_4 (x_1^2 + x_2^2), \quad (10c)
$$

$$
\dot{y}_4 = \frac{1}{4} \sigma_2 y_3 + \frac{3}{4} \beta_1 y_3^3 + \frac{1}{2} \beta_2 y_3 y_3^2 + \frac{3}{4} \beta_1 y_3 y_4^2. \quad (10d)
$$

The normal form with parameters can be written as

$$
\dot{y}_1 = -\overline{\mu} y_1 + (1 - \overline{\sigma}_1) y_2, \quad (11a)
$$

$$
\dot{y}_2 = \overline{\sigma}_1 y_1 - \overline{\sigma}_2 y_2 + \frac{3}{2} \alpha_1 y_1^3 + \alpha_2 y_1 y_2^2 + \alpha_2 y_1 y_4^2, \quad (11b)
$$

$$
\dot{y}_3 = -\overline{\mu} y_3 - \overline{\sigma}_2 y_4 - \frac{3}{4} \beta_1 y_4 (x_3^2 + x_4^2) - \frac{1}{2} \beta_2 y_4 (x_1^2 + x_2^2), \quad (11c)
$$

$$
\dot{y}_4 = \overline{f}_2 + \overline{\sigma}_2 y_3 - \overline{\mu} y_4 + \frac{3}{4} \beta_1 y_3^3 + \frac{1}{2} \beta_2 y_3 y_3^2 + \frac{3}{4} \beta_1 y_3 y_4^2, \quad (11d)
$$

where $\overline{\mu} = \frac{1}{2} \mu$, $\overline{\sigma}_2 = \frac{1}{4} \sigma_2$ and $\overline{f}_2 = \frac{1}{8} F_2$.

Further, we let

$$
y_3 = I \cos \gamma \quad \text{and} \quad y_4 = I \sin \gamma. \quad (12)
$$

Substituting equation (12) into equation (11) yields

$$
\dot{y}_1 = -\overline{\mu} y_1 + (1 - \overline{\sigma}_1) y_2, \quad (13a)
$$

$$
\dot{y}_2 = \overline{\sigma}_1 y_1 - \overline{\sigma}_2 y_2 + \frac{3}{2} \alpha_1 y_1^3 + \alpha_2 y_1 y_2^2 + \alpha_2 y_1 y_4^2, \quad (13b)
$$

$$
\dot{I} = -\overline{\mu} I - \overline{f}_2 \sin \gamma, \quad (13c)
$$

$$
\dot{\gamma} = \overline{\sigma}_2 I + \frac{3}{4} \beta_1 I^3 + \frac{1}{2} \beta_2 I y_1^2 - \overline{f}_2 \cos \gamma. \quad (13d)
$$

In order to get the unfolding of equation (13), a linear transformation is introduced as

$$
\begin{bmatrix}
  y_1 \\
  y_2 \\
\end{bmatrix} = \begin{bmatrix}
  \sqrt{\alpha_2} & 1 - \overline{\sigma}_1 & 0 & u_1 \\
  \sqrt{\frac{1}{2} \beta_2} & 1 - \overline{\sigma}_1 & \overline{\mu} & 1 & u_2 \\
\end{bmatrix}. \quad (14)
$$

Then, we have

$$
\begin{bmatrix}
  u_1 \\
  u_2 \\
\end{bmatrix} = \begin{bmatrix}
  \sqrt{\frac{1}{2} \beta_2} & 1 & 0 & \sqrt{\frac{1}{2} \beta_2} & 1 & 0 \\
  (1 - \overline{\sigma}_1) & \overline{\mu} & 1 - \overline{\sigma}_1 & y_1 & y_2 \\
\end{bmatrix}. \quad (15)
$$
Substituting equations (14) and (15) into equation (13) and canceling nonlinear terms which include the parameter \( \sigma_1 \) yield the unfolding as

\[
\begin{align*}
\dot{u}_1 &= u_2, \quad (16a) \\
\dot{u}_2 &= -\mu_1 u_1 - \mu_2 u_2 + \eta_1 u_1^3 + \alpha_2 u_1 I^2, \quad (16b) \\
I &= -\tilde{\mu} I - \tilde{f}_2 \sin \gamma, \quad (16c) \\
F_l &= \sigma_2 I + \eta_2 I^3 + \alpha_2 u_1^2 - \tilde{f}_2 \cos \gamma, \quad (16d)
\end{align*}
\]

where \( \mu_1 = \tilde{\mu}^2 - \sigma_1 (1 - \sigma_1) \), \( \mu_2 = 2 \tilde{\mu}_1 \), \( \eta_1 = \frac{3}{4} \alpha_2 \), and \( \eta_2 = \frac{3}{4} \beta_1 \).

The scale transformations may be introduced as follows

\[
\begin{align*}
\mu_2 &\to \epsilon \mu_2, \quad \tilde{\mu} \to \epsilon \tilde{\mu}, \quad \tilde{f}_2 \to \epsilon \tilde{f}_2, \quad \eta_1 \to \eta_1, \quad \eta_2 \to \eta_2, \\
\end{align*}
\]

Then, normal form (16) can be rewritten as the form with the perturbations

\[
\begin{align*}
\dot{u}_1 &= \frac{\partial H}{\partial u_2} + \epsilon g^u = u_2, \quad (18a) \\
\dot{u}_2 &= -\frac{\partial H}{\partial u_1} + \epsilon g^\mu = -\mu_1 u_1 + \eta_1 u_1^3 + \alpha_2 u_1 I^2 - \epsilon \mu_2 u_2, \quad (18b) \\
\dot{I} &= \frac{\partial H}{\partial I} + \epsilon g^I = -\epsilon \tilde{\mu} I - \epsilon \tilde{f}_2 \sin \gamma, \quad (18c) \\
F_l &= -\frac{\partial H}{\partial I} + \epsilon g^\gamma = \sigma_2 I + \eta_2 I^3 + \alpha_2 u_1^2 - \epsilon \tilde{f}_2 \cos \gamma, \quad (18d)
\end{align*}
\]

where the Hamiltonian function \( H \) is of the form

\[
H(u_1, u_2, I, \gamma) = \frac{1}{2} u_2^2 + \frac{1}{2} \tilde{\mu} u_1^2 - \frac{1}{4} \eta_1 u_1^4 - \frac{1}{2} \alpha_2 I^2 u_1^2 - \frac{1}{2} \sigma_2 I^2 - \frac{1}{4} \eta_2 I^4,
\]

and \( g^u \), \( g^\mu \), \( g^I \) and \( g^\gamma \) are the perturbation terms induced by the dissipative effects

\[
g_0^u = 0, \quad g_0^\mu = -\mu_2 u_2, \quad g_1^I = -\tilde{\mu} I - \tilde{f}_2 \sin \gamma, \quad g_1^\gamma = \tilde{f}_2 \cos \gamma.
\]

### 4. Unperturbed Dynamics

When \( \epsilon = 0 \), it is noted that system (18) is an uncoupled two-degree-of-freedom nonlinear system. The \( I \) variable appears in \((u_1, u_2)\) components of system (18) as a parameter since \( I = 0 \). Consider the first two decoupled equations

\[
\begin{align*}
\dot{u}_1 &= u_2, \quad (21a) \\
\dot{u}_2 &= -\mu_1 u_1 + \eta_1 u_1^3 + \alpha_2 u_1 I^2 u_1, \quad (21b)
\end{align*}
\]

Since \( \eta_1 > 0 \), system (21) can exhibit the heteroclinic bifurcations. It is easy to see from equation (21) that when \( \mu_1 - \alpha_2 I^2 < 0 \), the only solution of system (21) is the trivial zero solution \((u_1, u_2) = (0, 0)\) which is the saddle point. On the curve defined by \( \mu_1 = \alpha_2 I^2 \), that is,

\[
\tilde{\mu}^2 = \sigma_1 (1 - \sigma_1) + \alpha_2 I^2,
\]

or

\[
I_{1,2} = \pm \left[ \frac{\tilde{\mu}^2 - \sigma_1 (1 - \sigma_1)}{\alpha_2} \right]^{\frac{1}{2}},
\]

the trivial zero solution may bifurcate into three solutions through a pitchfork bifurcation, which are given by \( q_0 = (0, 0) \) and \( q_{\pm}(I) = (B, 0) \), respectively, where

\[
B = \pm \left( \frac{1}{\eta_1 \left[ \tilde{\mu}^2 - \sigma_1 (1 - \sigma_1) - \alpha_2 I^2 \right]} \right)^{\frac{1}{2}}.
\]
From the Jacobian matrix evaluated at the non-zero solutions, it is known that the singular points \( q_\pm(I) \) are the saddle points.

It is observed that \( I \) and \( \gamma \) may represent actually the amplitude and phase of the vibrations. Therefore, we may assume that \( I \geq 0 \) and equation (23) becomes

\[
I_1 = 0 \quad \text{and} \quad I_2 = \left[ \frac{\mu^2 - \bar{\sigma}_1(1 - \bar{\sigma}_1)}{\alpha_2} \right]^{\frac{1}{2}}, \tag{25}
\]

such that for all \( I \in [I_1, I_2] \), system (21) has two hyperbolic saddle points, \( q_\pm(I) \), which are connected by a pair of heteroclinic orbits, \( u^h_{\pm}(T_1, I) \), that is, \( \lim_{T_1 \to \pm \pi} u^h_{\pm}(T_1, I) = q_\pm(I) \). So in full four-dimensional phase space the set defined by

\[
M = \{ (u, I, \gamma) | u = q_\pm(I), I_1 \leq I \leq I_2, 0 \leq \gamma \leq 2\pi \} \tag{26}
\]

is a two-dimensional invariant manifold. From the results obtained in reference [9], it is known that the two-dimensional invariant manifold \( M \) is normally hyperbolic. The two-dimensionally normally hyperbolic invariant manifold \( M \) has three-dimensional stable and unstable manifolds which are represented as \( W^s(M) \) and \( W^u(M) \) respectively. The existence of the heteroclinic orbit of system (43) to \( q_\pm(I) = (B, 0) \) indicates that \( W^s(M) \) and \( W^u(M) \) intersect non-transversally along a three-dimensional heteroclinic manifold denoted by \( \Gamma \), which can be written as

\[
\Gamma = \left\{ (u, I, \gamma) | u = u^h_{\pm}(T_1, I), I_1 < I < I_2, \gamma = \int_{0}^{T_1} D_I H(u^h_{\pm}(T_1, I), I) ds + \gamma_0 \right\}. \tag{27}
\]

Now we analyze the dynamics of the unperturbed system of equations (18) restricted to \( M \). Considering the unperturbed system of equation (18) restricted to \( M \) yields

\[
I = 0, \quad \Gamma_I = D_I H(q_\pm(I), I), \quad I_1 \leq I \leq I_2, \tag{28}
\]

where

\[
D_I H(q_\pm(I), I) = -\frac{\partial H(q_\pm(I), I)}{\partial I} = \bar{\sigma}_2 I' + \eta_2 I'^3 + \alpha_2 q_\pm^2(I). \tag{29}
\]

From the results obtained by Kovacic and Wiggins [8,9], it is known that if \( D_I H(q_\pm(I), I) \neq 0 \) then \( I = \text{constant} \) is called as a periodic orbit and if \( D_I H(q_\pm(I), I) = 0 \) then \( I = \text{constant} \) is called as a circle of the singular points. A value of \( I \in [I_1, I_2] \) at which \( D_I H(q_\pm(I), I) = 0 \) is called as a resonant \( I \) value and these singular points as resonant singular points. We denoted a resonant value by \( I_\gamma \), so that

\[
D_I H(q_\pm(I), I) = \bar{\sigma}_2 I' + \eta_2 I'^3 + \frac{\alpha_2}{\eta_1} [\bar{\sigma}_1 (1 - \bar{\sigma}_1) - \alpha_2 I'^2] I_\gamma = 0. \tag{30}
\]

Then, we obtain

\[
I_\gamma = \pm \left( \frac{\bar{\sigma}_1 \eta_1 + \alpha_2 (\bar{\sigma}_1 (1 - \bar{\sigma}_1))}{\alpha_2^2 - \eta_1 \eta_2} \right)^{\frac{1}{2}}. \tag{31}
\]

The geometry structure of the stable and unstable manifolds of \( M \) in full four-dimensional phase space for the unperturbed system of equation (18) is given in Figure 2. Because \( \gamma \) may represent the phase of the oscillations, when \( I = I_\gamma \), the phase shift \( \Delta \gamma \) of the oscillations is defined as

\[
\Delta \gamma = \gamma(\pm \infty, I_\gamma) - \gamma(-\infty, I_\gamma). \tag{32}
\]

The physical interpretation of the phase shift is the phase difference between the two end points of the orbit. In \((u_1, u_2)\) subspace, there exist a pair of the heteroclinic orbits connecting to the two saddles. Therefore, in fact the homoclinic orbit in \((I, \gamma)\) subspace is of a heteroclinic connecting in full four-dimensional space \((u_1, u_2, I, \gamma)\). The phase shift may denote the difference of \( \gamma \) value as a trajectory leaves and returns to the basin of attraction of \( M \). We will use the phase shift in subsequent analysis to obtain the condition for the existence of the multi-pulse Shilnikov-type homoclinic orbit. The phase shift will be calculated in the later analysis given for the heteroclinic orbit.
We consider the heteroclinic orbits of system (21). Letting \( \varepsilon_1 = \mu_1 - \alpha_2 T^2 \) and \( \mu_2 = \varepsilon_2 \), system (21) can be rewritten as

\[
\begin{align*}
\dot{u}_1 &= u_2, \quad \text{(33a)} \\
\dot{u}_2 &= -\varepsilon_1 u_1 + \eta_1 u_1^3 - \varepsilon_2 u_2.
\end{align*}
\]

Setting \( \varepsilon = 0 \) in equation (33), we see that system (33) is a Hamiltonian system with Hamiltonian

\[
H(u_1, u_2) = \frac{1}{2} u_1^2 + \frac{1}{2} \varepsilon_1 u_1^2 - \frac{1}{4} \eta_1 u_1^4.
\] (34)

When \( H = \frac{\varepsilon_1^2}{4\eta_1} \), there exists a heteroclinic loop \( \Gamma^0 \) which consists of the two hyperbolic saddles \( q_s \) and a pair of heteroclinic orbits \( u_s(T_1) \). In order to calculate the phase shift and the energy difference function, we need to obtain the equations of a pair of heteroclinic orbits which are given as

\[
\begin{align*}
u_1(T_1) &= \pm \sqrt{\frac{\varepsilon_1}{\eta_1} \tanh \left( \frac{\sqrt{2\varepsilon_1}}{2} T_1 \right)}, \quad \text{(35a)} \\
u_2(T_1) &= \pm \sqrt{\frac{\varepsilon_1}{2\eta_1} \sech \left( \frac{\sqrt{2\varepsilon_1}}{2} T_1 \right)}, \quad \text{(35b)}
\end{align*}
\]

Let us turn our attention to the computation of the phase shift. Substituting the first equation of equation (35) into the fourth equation of the unperturbed system of equation (18) yields

\[
\dot{\gamma} = \sigma_2 + \eta_2 T^2 + \frac{\alpha_2 \varepsilon_1}{\eta_1} \tanh \left( \frac{\sqrt{2\varepsilon_1}}{2} T_1 \right).
\] (36)

Integrating equation (36) yields

\[
\gamma(T_1) = \omega_r T_1 - \frac{\alpha_2 \sqrt{2\varepsilon_1}}{\eta_1} \tan \left( \frac{\sqrt{2\varepsilon_1}}{2} T_1 \right) + \gamma_0,
\] (37)

where \( \omega_r = \sigma_2 + \eta_2 T^2 + \frac{\varepsilon_1 \alpha_2}{\eta_1} \).
Figure 2 The geometric structure of manifolds $M$, $W^s(M)$ and $W^u(M)$ in full four-dimensional phase space.

At $I = I_r$, there is $\omega_r = 0$. Therefore, the phase shift may be expressed as

$$\Delta \gamma = \left[ -\frac{2\alpha_2 \sqrt{2\epsilon_1}}{\eta_1} \right]_{I=I_r} = -\frac{2\alpha_2}{\eta_1} \sqrt{2(\mu_1^2 - \sigma_1(1 - \sigma_1) - \alpha_2 I_r^2)}.$$  

(38)

5. Existence of Multi-Pulse Homoclinic Orbits

After obtaining detailed information on the nonlinear dynamic characteristics of $(u_1, u_2)$ components for the unperturbed system of (18), the next step is to examine the effects of small perturbation terms ($0 < \epsilon << 1$) on the unperturbed system of (18). In this section, the extended Melnikov method presented by Camassa and Kovacic [11] is employed to analyze the multi-pulse Shilnikov orbits and chaotic dynamics in the nonlinear oscillations of the simply supported rectangular thin plate subjected to transversal and in-plane excitations. We start by studying the influence of such small perturbations on the manifold $M$. The objective of research is to identify the parameter regions where the existence of the multi-pulse orbits is possible in the perturbed phase space. These orbits, which are negatively asymptotic to some invariant manifold in the slow manifold $M_{\epsilon}$, leave and enter a small neighborhood of $M_{\epsilon}$ multiple times, and then finally return and approach an invariant set in $M_{\epsilon}$ asymptotically. It will be indicated that these multi-pulse orbits can result from the Hamiltonian as well as dissipative perturbations. The existence of such multi-pulse orbits provides a robust mechanism for the existence of the complicated dynamics in the perturbed system. In this section, the emphasis is put on the application aspects of the extended Melnikov method to equation (18).
5.1 Dissipative Perturbations

We analyze the dynamics of the perturbed system and the influence of small perturbations on $M$. Based on the analysis in references [8-11], we know that $M$ along with its stable and unstable manifolds are invariant under small, sufficiently differentiable perturbations. It is noticed that $q_\pm(I)$ may persist under small perturbations, in particular, $M \rightarrow M_c$.

Therefore, we obtain

$$M = M_c = \{(u, I, \gamma) | u = q_\pm(I), I_1 \leq I \leq I_2, 0 \leq \gamma < 2\pi\}. \tag{39}$$

Considering the later two equations of system (18) yields

$$\dot{I} = -\overline{\mu} I - \tilde{f}_2 \sin \gamma, \tag{40a}$$
$$\dot{\gamma} = \alpha_2 I^2 + \sigma_2 u_1^2 - \frac{\tilde{f}_2}{I} \cos \gamma. \tag{40b}$$

It is known from the aforementioned analysis that the last two equations of system (18) are of a pair of pure imaginary eigenvalues. Therefore, the resonance can occur in system (40). Also introduce the scale transformations

$$\overline{\mu} \rightarrow \epsilon \overline{\mu}, \ I = I_r + \sqrt{\epsilon} h, \ \tilde{f}_2 \rightarrow \epsilon \tilde{f}_2, \ T_1 \rightarrow \frac{T_1}{\sqrt{\epsilon}}. \tag{41}$$

Substituting the above transformations into equations (40) yields

$$\dot{h} = -\overline{\mu} I_r - \tilde{f}_2 \sin \gamma - \sqrt{\epsilon} h \overline{\mu}, \tag{42a}$$
$$\dot{\gamma} = -\frac{2\delta}{\eta_1} I_r h - \sqrt{\epsilon} (\delta h^2 + \frac{\tilde{f}_2}{I_r} \cos \gamma), \tag{42b}$$

where $\delta = \alpha_2^2 - \eta_1 \eta_2$. When $\epsilon = 0$, equation (42) becomes

$$\dot{h} = -\overline{\mu} I_r - \tilde{f}_2 \sin \gamma, \quad \tag{43a}$$
$$\dot{\gamma} = -\frac{2\delta}{\eta_1} I_r h. \quad \tag{43b}$$

The unperturbed system (43) is a Hamiltonian system with the Hamiltonian function

$$\tilde{H}_\theta (h, \gamma) = -\overline{\mu} I_r \gamma + \tilde{f}_2 \cos \gamma + \frac{\delta}{\eta_1} I_r h^2. \tag{44}$$

The singular points of system (43) are given as

$$p_0 = (0, \gamma_s) = \left\{0, -\arcsin \frac{\overline{\mu} I_r}{\tilde{f}_2}\right\} \quad \text{and} \quad q_0 = (0, \gamma_s) = \left\{0, \pi + \arcsin \frac{\overline{\mu} I_r}{\tilde{f}_2}\right\}. \tag{45}$$

Based on the characteristic equations evaluated at the two singular points $p_0$ and $q_0$, we can know the stabilities of these singular points. The Jacobian matrix of equation (43) is

$$J = \begin{bmatrix} 0 & -\tilde{f}_2 \cos \gamma \\ -\frac{2\delta}{\eta_1} I_r & 0 \end{bmatrix}. \tag{46}$$

The characteristic equation corresponding to the singular point $p_0$ is obtained as

$$\lambda^2 - \frac{2\delta}{\eta_1} I_r \tilde{f}_2 \cos \gamma = 0. \tag{47}$$

When the condition $\frac{2\delta}{\eta_1} I_r \tilde{f}_2 \cos \gamma < 0$ is satisfied, equation (43) has a pair of pure imaginary eigenvalues. Therefore, it is known that the singular point $p_0$ is a center.

The characteristic equation corresponding to the singular point $q_0$ is obtained as

$$\lambda^2 - \frac{2\delta}{\eta_1} I_r \tilde{f}_2 \cos \gamma = 0. \tag{48}$$
When the condition \( \frac{2\delta}{\eta_0} I_r \tilde{f}_2 \cos \gamma_s > 0 \) is satisfied, equation (43) has two real, unequal and opposite sign eigenvalues. Therefore, the singular point \( q_0 \) is a saddle which is connected to itself by a homoclinic orbit. The phase portrait of equation (43) is given in Figure 3(a).

It is found that for sufficiently small \( \varepsilon \), the singular point \( q_0 \) remains a hyperbolic singular point \( q_c \) of saddle stability type. It is known that the Jacobian matrix of the linearization of equation (42) is of the form

\[
J_{p_c} = \begin{bmatrix} -\sqrt{\varepsilon \mu} & -\tilde{f}_2 \cos \gamma_c \\ -\frac{2\delta}{\eta_0} I_r & \sqrt{\varepsilon \mu} \end{bmatrix}.
\]

or

\[
J_{p_c} = \begin{bmatrix} -\sqrt{\varepsilon \mu} & -\tilde{f}_2 \cos \gamma_c \\ -\frac{2\delta}{\eta_0} I_r & -\sqrt{\varepsilon \mu} \end{bmatrix}.
\]

Based on equation (49), we find that the leading order term of the trace in the linearization of equation (42) is less than zero inside the homoclinic loop. Therefore, for the small perturbations, the singular point \( p_0 \) becomes a hyperbolic sink \( p_c \). The phase portrait of perturbed system (42) is also depicted in Figure 3(b).

![Figure 3](image)

**Figure 3**  Dynamics on the normally hyperbolic manifold; (a) the unperturbed case; (b) the perturbed case

At \( h = 0 \), the estimate of basin of attractor for \( \gamma_{\min} \) is obtained as

\[
-\pi I_r \gamma_{\min} + \tilde{f}_2 \cos \gamma_{\min} = -\pi I_r \gamma_s + \tilde{f}_2 \cos \gamma_s.
\]

Substituting \( \gamma_s \) in equation (45) into equation (51) yields

\[
\gamma_{\min} - \frac{\tilde{f}_2}{\mu I_r} \cos \gamma_{\min} = \pi + \arcsin \frac{\mu I_r}{\tilde{f}_2} + \frac{\sqrt{\mu^2 - \tilde{f}_2^2}}{\mu I_r}.
\]

Define an annulus \( A_c \) near \( I = I_c \) as

\[
A_c = \left\{ (u_1, u_2, I, \gamma) \mid |u_1 - B, u_2 = 0, |I - I_c| < \sqrt{\varepsilon C}, \gamma \in \mathbb{T}^{d} \right\},
\]

where \( C \) is a constant, which is chosen sufficient large so that the unperturbed homoclinic orbit is enclosed within the annulus. It is noticed that three-dimensional stable and unstable manifolds of \( A_c \), denoted as \( W^s(A_c) \) and \( W^u(A_c) \), are subsets of \( W^s(M_c) \) and \( W^u(M_c) \), respectively. We will indicate that for the perturbed system, the saddle focus \( p_c \) on \( A_c \) has the multi-pulse homoclinic orbits which
come out of the annulus $A_c$ and can return to the annulus in full four-dimensional space, and eventually may give rise to the multi-pulse Shilnikov type homoclinic loop, as shown in Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{The Shilnikov type three-pulse homoclinic orbits}
\end{figure}

5.2 The k-pulse Melnikov Function

We use a general method for finding heteroclinic orbits that make several consecutive fast excursions away from a set of hyperbolic manifolds by constructing an extension of the Melnikov method. We reduce the search for multi-pulse heteroclinic excursions to that of finding non-degenerate zeros of a function, $M_k(\varepsilon, I, \gamma_0, \mu)$, of certain parameters $\varepsilon, I, \gamma_0, \mu$, which we call the $k$-pulse Melnikov function. This function is computed by a recursion procedure from the ordinary 1-pulse Melnikov function, and depends on the small perturbation parameter $\varepsilon$, which is at variance with the usual Melnikov method and is peculiar to the general case of fast dynamics on the hyperbolic manifold. Moreover, the dependence on $\varepsilon$ is through a logarithmic function, which makes the calculation of the asymptotics in the small $\varepsilon$ limit particularly delicate.

We apply the extended Melnikov method to multi-pulse orbits heteroclinic to resonance bands. We use the new Melnikov method for extending the results to cover heteroclinic orbits with several consecutive fast pulses rather than just one. This is a typical singular perturbation problem in which there are two different time scales, and the dynamics on the hyperbolic manifold is slow. Homoclinic orbits are constructed by concatenating pieces of slow-time orbits on the hyperbolic manifold and fast-time heteroclinic orbits off of this manifold. Since the motion along the hyperbolic manifold is slow in this problem, this theory simplifies considerably due to the facts that the $k$-pulse Melnikov function does not depend on the small parameter $\varepsilon$, and that the non-folding condition is automatically satisfied and thus not needed. The $k$-pulse Melnikov function in this case becomes identical to the energy-phase function of references [10,11], which, however, was derived in an entirely different fashion. Moreover, there is a conceptual difference between two approaches.

The $k$-pulse Melnikov function is the geometric interpretation of a signed distance measured along the normal to a heteroclinic manifold which replaces the estimate of the change of energy computed along unperturbed heteroclinic orbits. On a more technical level, the construction of the energy-phase function developed in [10,11] crucially employs the details of the geometry that depends on the dynamics along the hyperbolic manifold being slow, while the derivation of the $k$-pulse Melnikov function avoids these details entirely at the price of the more delicate local estimates near the hyperbolic manifold.

In order to show the existence of multi-pulse heteroclinic orbits, it is important to obtain the expression of the $k$-pulse Melnikov function. Firstly, we computed 1-pulse Melnikov function based on equation (18) at the resonance $I = I_r$. The Melnikov function $M(I, \gamma_0, \mu)$ on both heteroclinic manifolds $W_+(M)$ and $W_-(M)$ is equal to
\[ M(I_r, \gamma_0, \bar{\mu}, \eta_1, \alpha_2, e_1) = \int_{-\infty}^{+\infty} \left( \frac{\partial}{\partial u_i} g_i^n + \frac{\partial}{\partial \theta_1} g_i^{\mu_1} + \frac{\partial}{\partial \theta_1} g_i^{e_1} + \frac{\partial}{\partial \theta_1} g_i^{e_1} \right) dT_i \]

\[ = -2\sqrt{2\mu_2 3\eta_1} e_1^{3/2} + \mu_2^2 \Delta \gamma - \tilde{f}_2 I_r \left[ \cos \left( \gamma_0 - \alpha_2 \frac{\sqrt{2e_1}}{\eta_1} \right) - \cos \left( \gamma_0 + \alpha_2 \frac{\sqrt{2e_1}}{\eta_1} \right) \right] \]

Then, the \( k \)-pulse Melnikov function is obtained as

\[ M_k(I_r, \gamma_0, \bar{\mu}, \eta_1, \alpha_2, e_1) = \sum_{j=0}^{k-1} M(I_r, \gamma_0 + j\Delta \gamma(I_r), \bar{\mu}, \eta_1, \alpha_2, e_1) \]

\[ = -\tilde{f}_2 I_r \left[ \cos \left( \gamma_0 - \alpha_2 \frac{\sqrt{2e_1}}{\eta_1} \right) - \cos \left( \gamma_0 + \alpha_2 \frac{\sqrt{2e_1}}{\eta_1} \right) \right] - 2\sqrt{2\mu_2 3\eta_1} e_1^{3/2} + \mu_2^2 \Delta \gamma \]

\[ - \tilde{f}_2 I_r \left[ \cos \left( \gamma_0 - \alpha_2 \frac{\sqrt{2e_1}}{\eta_1} - 2(k-1)\alpha_2 \frac{\sqrt{2e_1}}{\eta_1} \right) - \cos \left( \gamma_0 + \alpha_2 \frac{\sqrt{2e_1}}{\eta_1} - 2(k-1)\alpha_2 \frac{\sqrt{2e_1}}{\eta_1} \right) \right] \]

\[ = -2\sqrt{2\mu_2 3\eta_1} e_1^{3/2} + \mu_2^2 \Delta \gamma + \ldots \]

\[ - \tilde{f}_2 I_r \left[ \cos \left( \gamma_0 - \alpha_2 \frac{\sqrt{2e_1}}{\eta_1} - 2(k-1)\alpha_2 \frac{\sqrt{2e_1}}{\eta_1} \right) - \cos \left( \gamma_0 + \alpha_2 \frac{\sqrt{2e_1}}{\eta_1} - 2(k-1)\alpha_2 \frac{\sqrt{2e_1}}{\eta_1} \right) \right] \]

\[ = -2\sqrt{2\mu_2 3\eta_1} e_1^{3/2} + \mu_2^2 \Delta \gamma - \tilde{f}_2 I_r \left[ \cos \left( \gamma_0 - \alpha_2 \frac{\sqrt{2e_1}}{\eta_1} - 2(k-1)\alpha_2 \frac{\sqrt{2e_1}}{\eta_1} \right) - \cos \left( \gamma_0 + \alpha_2 \frac{\sqrt{2e_1}}{\eta_1} - 2(k-1)\alpha_2 \frac{\sqrt{2e_1}}{\eta_1} \right) \right] \]

\[ = -2\sqrt{2\mu_2 3\eta_1} e_1^{3/2} + \mu_2^2 \Delta \gamma - \tilde{f}_2 I_r \left[ \cos \left( \gamma_0 - \alpha_2 \frac{\sqrt{2e_1}}{\eta_1} - 2(k-1)\alpha_2 \frac{\sqrt{2e_1}}{\eta_1} \right) - \cos \left( \gamma_0 + \alpha_2 \frac{\sqrt{2e_1}}{\eta_1} - 2(k-1)\alpha_2 \frac{\sqrt{2e_1}}{\eta_1} \right) \right] \]

\[ If \ we \ set \ \Delta \gamma = -2\alpha_2 \frac{\sqrt{2e_1}}{\eta_1}, \ \gamma_{k-1} = \gamma_0 + (k-1)\frac{\Delta \gamma}{2}, \ this \ formula \ can \ be \ rewritten \ as \]

\[ M_k(I_r, \gamma_0, \bar{\mu}, \eta_1, \alpha_2, e_1) = M_k(I_r, \gamma_{k-1} - (k-1)\frac{\Delta \gamma}{2}, \bar{\mu}, \eta_1, \alpha_2, e_1) \]

\[ = \tilde{f}_2 I_r \left[ \cos \left( \gamma_{k-1} - \frac{1}{2}k\Delta \gamma \right) - \cos \left( \gamma_{k-1} + \frac{1}{2}k\Delta \gamma \right) \right] + \frac{k\mu_2 e_1}{3\alpha_2} \Delta \gamma + 2\mu_2^2 \left( \frac{1}{2}k\Delta \gamma \right) \]

\[ = 2\tilde{f}_2 I_r \left[ \sin \left( \gamma_{k-1} \right) \sin \left( \frac{1}{2}k\Delta \gamma \right) + \frac{1}{3\alpha_2} \frac{2e_1}{\eta_1} \left( \frac{1}{2}k\Delta \gamma \right) + 2\mu_2 I_r \left( \frac{1}{2}k\Delta \gamma \right) \right]. \]
There are two simple zeroes of the \( k \)-pulse Melnikov function in the interval \( \gamma_{k-1} \in [0, \pi] \), that is

\[
\tilde{\gamma}_{k-1,1} = -\arcsin \left( \frac{1}{2} k \Delta \gamma \right) \left( \frac{\mu_1 e_1 + 3 \alpha_2 \mu_2^2}{\sin \left( \frac{1}{2} k \Delta \gamma \right) (3 \alpha_2 f_2 I_2)} \right),
\]

\( (61) \)

\[
\tilde{\gamma}_{k-1,2} = \pi + \tilde{\gamma}_{k-1,1}.
\]

\( (62) \)

Then the \( k \)-pulse Melnikov function (56) has simple zeroes in \( \gamma_{k-1} \) at some \( \gamma_{k-1} = \tilde{\gamma}_{k-1,1} \) and \( \gamma_{k-1} = \tilde{\gamma}_{k-1,2} = \pi + \tilde{\gamma}_{k-1,1} \). If for \( i = 1 \) or \( i = 2 \), the values of the \( j \)-pulse Melnikov function \( M_j (I_r, \tilde{\gamma}_{0,i}, \mu_1, \eta_1, \alpha_2, e_1) \) are different from zero, and if \( k, \mu_1, \eta_1, \alpha_2, \) and \( e_1 \) satisfy the inequality (58) and equation (59), then the stable and unstable manifolds \( W^s (M_k) \) and \( W^u (M_k) \) intersect transversely along a symmetric pair of two-dimensional, \( k \)-pulse singular surfaces \( \sum_{\epsilon, z, \epsilon_1} \left( \tilde{\gamma}_{k,1,i} \right) \). In the phase space of the unperturbed system (18), this pair collapses smoothly onto a pair of limiting \( k \)-pulse surfaces, \( \sum_{\epsilon, 0} \left( \tilde{\gamma}_{k,1,i} \right) \), parametrized by the expressions (35) and (37) with \( I = I_r \), \( \gamma_0 = \tilde{\gamma}_{k-1,i} - (k-1) \frac{1}{2} \Delta \gamma + j \Delta \gamma \), where \( j = 0, \ldots, k-1 \), and arbitrary \( h \). The sign in each of these expressions is determined by the sign of the corresponding \( j \)-pulse Melnikov function \( M_j (I_r, \tilde{\gamma}_{0,j}, \mu_1, \eta_1, \alpha_2, e_1) \).

From the discussion in the previous paragraph, it easily follows that for \( \gamma_{0,i} = \tilde{\gamma}_{k-1,i} - (k-1) \frac{1}{2} \Delta \gamma + j \Delta \gamma \), with \( i = 1, 2 \), the values of the \( j \)-pulse Melnikov functions \( M_j (I_r, \tilde{\gamma}_{0,i}, \mu_1, \eta_1, \alpha_2, e_1) \) are different from zero for all \( j = 1, \ldots, k-1 \), and are in fact of the same sign for all \( j \). This sign is negative for \( \tilde{\gamma}_{0,1} \) and positive for \( \tilde{\gamma}_{0,2} \). Therefore, the \( k \)-pulse singular surfaces \( \sum_{\epsilon, z, \epsilon_1} \left( \tilde{\gamma}_{k,1,i} \right) \) and \( \sum_{\epsilon, z, \epsilon_1} \left( \tilde{\gamma}_{k,1,2} \right) \) indeed exist in this case for all \( k \), and so do the limiting \( k \)-pulse surfaces, \( \sum_{\epsilon, 0} \left( \tilde{\gamma}_{k,1,i} \right) \) and \( \sum_{\epsilon, 0} \left( \tilde{\gamma}_{k,1,2} \right) \). Since the regions enclosed by the two heteroclinic manifolds \( W_s (M) \) and \( W_u (M) \) are both convex, and the normal \( n = (\mu_1 u_1 + \eta_1 u_1^2 + \alpha_2 f_2 u_1, -u_2, 0, 0) \) is easily seen to point out of them, it follows that orbits forming each of the surfaces \( \sum_{\epsilon, 0} \left( \tilde{\gamma}_{k,1,1} \right) \) are parametrized by expressions (35) and (37) with alternating signs, and orbits forming each of the surfaces \( \sum_{\epsilon, 0} \left( \tilde{\gamma}_{k,1,2} \right) \) are parametrized by expressions (35) and (37) with the same signs. Figure 5 shows \( 2 \)-pulse singular surfaces.

From the discussion in the previous two paragraphs, it follows that there exist an integer \( n \geq 1 \) and \( 2n \) symmetric pairs of \( k \)-pulse singular orbits \( \Gamma_{zk}^k \), with \( k = 1, \ldots, n \), which lie on the \( n \) pairs of limiting intersection surfaces \( \sum_{\epsilon, 0} \left( \tilde{\gamma}_{k,1,1} \right) \), and connect \( 2n \) pairs of points on the separatrix on the cylinder \( M_0 \). The sign in the subscript of the symbol \( \Gamma_{zk}^k \) is the same as the sign of the corresponding surfaces \( \sum_{\epsilon, 0} \left( \tilde{\gamma}_{k,1,1} \right) \); the sign in the superscript is the same as the sign of the \( h \) coordinate along the singular orbit \( \Gamma_{zk}^k \). As mentioned above, the \( u_1 \) coordinates along the pulses of the orbit \( \Gamma_{zk}^k \) have alternating signs. The equality of the \( k \)-pulse Melnikov function and the corresponding difference in the values of the system (44) implies that for \( k > j \), the takeoff point of the singular orbit \( \Gamma_{zk}^k \) is to the right of the takeoff point of the singular orbit \( \Gamma_{zj}^j \), and the landing point of the singular orbit \( \Gamma_{zk}^k \) is to the
left of the landing point of the singular orbit $\Gamma_{x,k}^+$. Moreover, the takeoff point of any singular orbit $\Gamma_{x,k}^+$ is to the right of the landing point of any other singular orbit $\Gamma_{x,l}^+$.

$$\pm \Gamma_l \pm \Delta \gamma \pm \gamma_{1,0} - \gamma \Delta - \gamma_{1,0} + \gamma$$

$$\pm \Gamma_k \pm \Delta \gamma \pm \gamma_{2,0} - \gamma \Delta - \gamma_{2,0} + \gamma$$

Figure 5  2-pulse singular surfaces

We can now form a countable infinity of singular heteroclinic orbits as follows. Each such orbit starts along the right-hand branch of the unstable manifold $W(q_0)$ of the saddle $q_0$ on the annulus $M_0$. The singular heteroclinic orbit then takes off from $M_0$ along one of the singular $k$-pulse orbits $\Gamma_{x,k}^+$, and lands back on $M_0$ at a point on the separatrix that connects the saddle $q_0$ to itself. After following the separatrix for a while, the singular heteroclinic orbit again takes off along some singular $l$-pulse orbit $\Gamma_{x,l}^+$, and so forth. Eventually, the singular heteroclinic orbit lands back on the separatrix and either follows it to the takeoff point of one of the two heteroclinic orbits $\Gamma_{x,j}^+$, where it takes another excursion along one of them before returning to the saddle $q_0$, or else proceeds directly to this saddle.

6. Numerical Results of Chaotic Motions

We choose averaged equation (8) to do the numerical simulations. Numerical approach through this computer software *Matlab* is utilized to explore the existence of the Shilnikov type multi-pulse chaotic motions in the nonlinear oscillations of the simply supported rectangular thin plate subjected to transversal and in-plane excitations.

When we respectively choose the transversal and in-plane excitations, parameters and initial conditions as $f_1 = 62$, $F_2 = 122.2$, $\mu = 0.03$, $\sigma_1 = 2.0$, $\sigma_2 = 3.5$, $\alpha_1 = -3.2$, $\alpha_2 = -5.1$, $\beta_1 = -2.7$, $\beta_2 = 6.3$, $x_{10} = 0.14$, $x_{20} = 0.55$, $x_{30} = 0.35$, $x_{40} = -0.180$, a form of chaotic response in the nonlinear oscillations of the simply supported rectangular thin plate is shown in Figure 6. Obviously, from the phase portrait in the three dimensional space $(x_1, x_2, x_3)$ in Figure 6, it is found that the Shilnikov type multi-pulse chaotic motion exists.
7 Conclusions

The Shilnikov type multi-pulse orbits and chaotic dynamics in the nonlinear oscillations of the simply supported rectangular thin plate subjected to transversal and in-plane excitations are investigated for the first time by using the analytical and numerical approaches when the averaged equations have one non-semisimple double zero and a pair of pure imaginary eigenvalues. The study is focused on co-existence of 1:2 internal resonance and primary parametric resonance-fundamental parametric resonance in equation (8). The extended Melnikov method is employed to demonstrate the existence of complex motions by identifying the existence of multi-pulse jumping orbits in the perturbed phase space for the dissipative perturbations. It is found from the aforementioned analytical investigation that the simply supported rectangular thin plate subjected to transversal and in-plane excitations can undergo the pitchfork bifurcation, Hopf bifurcation, heteroclinic bifurcations and the Shilnikov type multi-pulse homoclinic orbits.

Based on the aforementioned analytical and numerical studies, it is found that the multi-pulse homoclinic orbits depend on dissipative perturbations and periodic excitations. We can show the
existence of $k$-pulse orbits homoclinic to the saddle $q_t$ that lie on the surfaces $\sum_{k=1}^{\infty} \left( \frac{1}{(k-1,k)} \right)^{\nu}$. The $\nu_1$ coordinate along all pulses of such an orbit has the same sign. It can be conjectured that the transfer of energy between the different modes occurs through the Shilnikov type multi-pulse homoclinic orbits.

In order to illustrate the theoretical predictions, the software is used to perform numerical simulation. The numerical results also show the existence of the Shilnikov type multi-pulse chaotic motions in the averaged equations. It is well known that the Shilnikov type multi-pulse chaotic motions in the averaged equations can lead to the Shilnikov type multi-pulse amplitude modulated chaotic oscillations in the original system under certain conditions. Therefore, it is demonstrated that there are the amplitude modulated chaotic motions of the multi-pulse Shilnikov type in the simply supported rectangular thin plate subjected to transversal and in-plane excitations.

Numerical simulations obtained in this paper indicate that there exist different shape of the chaotic responses in the nonlinear oscillations of the rectangular thin plate under certain transversal and in-plane excitations, parameters and initial conditions. It is found from numerical simulations that the shape of the chaotic motions is completely different. We also find that the transversal excitations $f_1$, and the in-plane excitation $F_2$ and damping coefficient $\mu$ have important influence on the chaotic motions in the nonlinear oscillations of the simply supported rectangular thin plate subjected to transversal and in-plane excitations.

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